Using path integrals to price interest rate derivatives

Matthias Otto

Institut für Theoretische Physik, Universität Göttingen
Bunsenstr. 9, D-37073 Göttingen

Abstract: We present a new approach for the pricing of interest rate derivatives which allows a direct computation of option premiums without deriving a (Black-Scholes type) partial differential equation and without explicitly solving the stochastic process for the underlying variable. The approach is tested by rederiving the prices of a zero bond and a zero bond option for a short rate environment which is governed by Vasicek dynamics. Furthermore, a generalization of the method to general short rate models is outlined. In the case, where analytical solutions are not accessible, numerical implementations of the path integral method in terms of lattice calculations as well as path integral Monte Carlo simulations are possible.

1 Introduction

The purpose of this article is to present a new approach for the pricing of interest rate derivatives: the path integral formalism. The claim is that interest rate derivatives can be priced without explicitly solving for the stochastic process of the underlying (e.g. a short rate) and without deriving a (Black-Scholes type) partial differential equation (PDE).

The mathematical foundation of the PDE approach to derivatives pricing, which is used traditionally, is the Feynman-Kac lemma [1, 2] connecting the solution of a certain type of parabolic PDE to expectation values with respect to stochastic processes. Usually, the original pricing problem is given in terms of an expectation value. Then the Feynman-Kac lemma is evoked in order to use the corresponding PDE to find the solution to the pricing problem. In the martingale approach to options pricing (see e.g.[3]), usually the stochastic process for the underlying is solved, and the option price is determined using stochastic calculus.

The path integral formalism widely used in theoretical physics and first introduced in this field by Feynman [1] does not need any reference to a Black-Scholes type equation nor does it depend on an explicit solution of the stochastic process for the underlying. It is traditionally introduced, however, by showing that a certain path integral satisfies a particular PDE which is known to describe the problem under investigation. This procedure is also followed by Linetsky [11] who reviewed the early application of path integrals to finance (initiated by Dash [12, 13]) and who discusses various analytically tractable payoffs . In statistical physics, the application of path integrals to option pricing has been recently discussed by Baaquie who derived a Schrödinger equation for the corresponding pricing problem [7]. Let us mention here, that topics in finance have become increasingly popular in theoretical physics (see e.g. Bouchaud [8] and references therein).

As opposed to the approach followed in [11, 7], in this article, we introduce the path integral from a stochastic viewpoint only (using the Martin-Siggia-Rose (MSR) formalism [14, 15, 16, 17]) without making any appeal to particular PDEs. In our view, this procedure is easier to carry out than proving that a specific path integral satisfies a particular PDE. The MSR method is well known in statistical physics and finds a rigorous justification in the context of the Cameron-Martin-Girsanov theorem from stochastic analysis [22].

The path integral approach is introduced here in the context of interest rate products. It has been discussed from a more general perspective by Linetsky [11]. In order to be specific, the framework is presented for the short rate model first introduced by Vasicek. The fact that analytical results exist for this model, permits a serious test of our approach. In particular, we value zero bonds and zero bond

options as special cases of path-dependant contingent claims.

In principle, every short rate model can be cast into a path integral. More generally, any process describing an underlying random variable (tradable or non-tradable) that can be modelled in terms of a Langevin equation (or in mathematical terms, a stochastic differential equation) can be rewritten in this approach. The path integral might itself serve as a starting point to state new models. The criterion of no-arbitrage that has to be fulfilled by any model (also by those presented in a path integral from) is slightly touched on in section 7.

The path integral approach is not just a new way to obtain results already known, but it may serve as a general framework for numerical work. In fact, one of the key ingredients is a probability density functional for one or several stochastic underlyings. A European claim is valued by integrating over all possible paths taken by the underlying variables according to the weighting imposed by the functional. Moreover, a Monte Carlo simulation can be set up by sampling the different paths taken by the underlying according to the probability density functional.

The article is divided as follows. In section 2 we discuss the Vasicek model as a prototype model that can be stated in terms of a Langevin equation. Next, in section 3 the pricing of contingent claims is considered in a general context. Then in section 4 we derive a probability density functional for the short rate which we then, in section 5, apply to calculate the price of a simple contingent claim, a zero bond, and in section 6, to determine the price of a zero bond option. In section 7, we make a few remarks on the path integral as a new guise for (short rate) models, before reaching a conclusion.

2 The Vasicek model as a Langevin equation

The Vasicek model [4] in finance is usually studied in terms of a stochastic differential equation for the short rate r, i.e. the rate of instantaneous borrowing (and lending, assuming no bid-offer spreads):

$$dr(t) = a(b - r(t))dt + \sigma dz(t) \tag{2.1}$$

The parameters a and b model the mean reversion, σ is the volatility of the short rate. Both a,b and σ are constant in time. The increment $dz(t)$ represents a Wiener process. The whole dynamics is parametrized in terms of the continuous time variable t. Next, we write the equivalent Langevin equation [10]:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = a(b - r(t)) + \sigma f(t) \tag{2.2}$$

Crudely speaking, this equation can be obtained by dividing Eq.(2.1) by dt. The stochastic "force" $f(t)$ is a practical way to think of the mathematical Wiener in-

crement dz per unit time dt, so very roughly, we say that

$$\frac{\mathrm{d}z}{\mathrm{d}t} \simeq f(t) \tag{2.3}$$

The Langevin equation (2.2) also describes the motion of Brownian particle which is attached by a spring.

The fact that dz is a Wiener process implies that the force $f(t)$ satisfies

$$\langle f(t)f(t')\rangle = \delta(t-t'), \quad \langle f(t)\rangle = 0 \tag{2.4}$$

The brackets $\langle \ldots \rangle$ indicate a mean value with respect to a Gaussian distribution of zero mean and variance of one.

3 Contingent claims

Using the terminology of [9], the so-called classical Vasicek approach to pricing contingent claims is to derive a PDE subject to appropriate boundary conditions which model the payoff profile. The PDE is obtained using Ito's lemma and setting up a riskless portfolio which eliminates the Wiener process. The solution of the PDE then gives the price for the claim at time t prior (and up) to maturity time T.

The Feynman-Kac lemma states that the solution of the particular PDE considered in the classical approach is given as an expectation value with respect to the martingale measure obtained from the stochastic process for the underlying tradable [10]. In our case this is the discounted bond price $Z(t,T) = B_t^{-1}P(t,T)$, where $P(t,T)$ is the (zero) bond price at time t promissing to pay one unit at maturity T and B_t is a cash bond at time t which has accrued form $B_0 = 1$ at time 0. The martingale measure for $Z(t,T)$ leads to a risk-adjusted process for the short rate $r(t)$, which in the case of the Vasicek model leads merely to a renormalization of the parameter ab. Later on in the discussion, we will assume that the parameters a and b are the risk-adjusted ones [9].

Now by virtue of the Feynman-Kac lemma the price of any claim $V(t)$ at time t whose payoff at time T is X is given by:

$$V(t) = E_Q\left[e^{-\int_t^T dsr(s)}X|r(t) = r\right] \tag{3.5}$$

The subscript Q indicates the martingale measure.

Later on, we will consider the bond price $P(t,T)$ as an example of a claim with payoff $X = 1$ at maturity T and an option on a zero bond both of which we will price using the path integral technique. Before, we set up the stage in order to be able to carry out the expectation value in Eq.(3.5).

4 The distribution functional for the short rate

We now provide the foundation for the path integral approach. In order to be specific, we will start from the stochastic process for the Vasicek model, or rather the corresponding Langevin equation Eq.(2.2). This equation is transformed using the so-called Martin-Siggia-Rose formalism [14, 15, 16, 17] to give a probability density functional for the short rate.

For a given realization of the force $f(t)$, we have the conditional probability density functional for all pathwise realizations of the short rate process given by Eq.(2.2):

$$p\left(\{r(s)\}|\{f(s)\}\right) = \mathcal{N}\prod_s \delta\left(\frac{\mathrm{d}r}{\mathrm{d}s} - a(b - r(s)) - \sigma f(s)\right) \tag{4.6}$$

The constant $\mathcal{N}$ is a normalization factor which is left unspecified. Of course, we later constrain the parameter s to range from t to T, as well as the product on the r.h.s of the equation. In what follows, this restriction $s \in [t, T]$ is understood. Next, we sum over the "force" $f(t)$:

$$p\left(\{r(s)\}\right) = \mathcal{N}\int \mathcal{D}f(s) e^{-\frac{1}{2}\int ds f^2(s)} \prod_s \delta\left(\frac{\mathrm{d}r}{\mathrm{d}s} - a(b - r(s)) - \sigma f(s)\right) \tag{4.7}$$

The integration is weighted by a Gaussian as we imposed white noise correlations (see Eq.(2.4)). Obviously, the measure $\mathcal{D}f(s)$ is a very peculiar one which we will comment on in more detail at the end of the next section. For the moment, let us consider it as a sum within the space of continuous real-valued functions $f(s)$.

Next we carry out the functional integral with respect to $f(s)$. For this purpose, the delta function needs to be rewritten for fixed s as follows:

$$\delta\left(\frac{\mathrm{d}r}{\mathrm{d}s} - a(b - r(s)) - \sigma f(s)\right) = \frac{1}{\sigma}\delta\left(\frac{1}{\sigma}\left(\frac{\mathrm{d}r}{\mathrm{d}s} - a(b - r(s))\right) - f(s)\right) \tag{4.8}$$

The factor $1/\sigma$ can be absorbed into the normalization constant because it is constant with respect to the parameter s. For time-dependant volatilities $\sigma = \sigma(s)$, which arise in general short rate models, the situation is more complicated (see section 8).

Finally, one obtains for the probability density functional for the short rate $r(s)$:

$$p\left(\{r(s)\}\right) = \mathcal{N} e^{-\frac{1}{2\sigma^2}\int ds\left(\frac{\mathrm{d}r(s)}{\mathrm{d}s} - a(b - r(s))\right)^2} \tag{4.9}$$

As mentioned, only the family of short rates within a specific time window is considered, e.g. between t and T. Therefore, the integration on the r.h.s. of Eq.(4.9) has to be constrained to the interval $[t, T]$. It is interesting to note here that negative short rates have a non-vanishing probability, a fact that is well-known for the Vasicek model. Alternative short rate models exist which bypass this drawback. In fact,

in the path integral approach one could also enforce non-negative short rates by a delta function which is a subject of future work. The probability density functional can now be used to carry out the expectation value for any claim in Eq.(3.5).

5 The bond price as a path integral

Let us discuss the bond price as a specific example for a claim. The bond price today, i.e. at time t, for a monetary unit promissed at some later time T can be stated as an expectation value of the discount function:

$$P(t,T) = E_Q\left[e^{-\int_t^T dsr(s)}|r(t)=r\right] \tag{5.10}$$

The classical Vasicek approach (as described in [9]) has been to write down the corresponding PDE for $P(t,T)$ rather than evaluating the expectation value in Eq.(5.10) directly.

Using the probability density functional in Eq.(4.9), we are now in the position to evaluate the expectation value in Eq.(5.10) for $P(t,T)$. It is explicitly given as follows:

$$P(t,T) = \frac{\int_{-\infty}^{\infty} dr(T)\int_{r(t)}^{r(T)}\mathcal{D}r(s)\exp\left(-\frac{1}{2\sigma^2}\int_t^T ds\left(\frac{\mathrm{d}r(s)}{\mathrm{d}s}-a(b-r(s))\right)^2-\int_t^T dsr(s)\right)}{\int_{-\infty}^{\infty} dr(T)\int_{r(t)}^{r(T)}\mathcal{D}r(s)\exp\left(-\frac{1}{2\sigma^2}\int_t^T ds\left(\frac{\mathrm{d}r(s)}{\mathrm{d}s}-a(b-r(s))\right)^2\right)} \tag{5.11}$$

The argument of the expectation value Eq.(5.10) appears in the numerator of the equation above as the second term inside the exponential function. The first term stems from the probability density functional Eq.(4.9). The denominator gives the proper normalization. The functional integrations in the numerator are due to the fact that a conditional expectation value on $r(t)=r$ is desired. Therefore, first a transition probability (modulo a constant) for the short rate to pass form $r(t)$ to $r(T)$ is calculated, and then an integration with respect to the final rate $r(T)$ is carried out.

Let us now substitute for $x(s)=b-r(s)$. Then after carrying out integrations involving boundary terms, Eq.(5.11) reads as

$$P(t,T) = \frac{X}{Y} \tag{5.12}$$

with

$$\begin{aligned} X &= \int_{-\infty}^{\infty} dx(T)\int_{x(t)}^{x(T)}\mathcal{D}x(s)\exp\left(-\frac{1}{2\sigma^2}\int_t^T ds\left(\left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)^2+a^2x^2(s)\right)\right. \\ &- \left.\frac{a}{2\sigma^2}(x^2(T)-x^2(t))-b(T-t)+\int_t^T dsx(s)\right) \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} Y &= \int_{-\infty}^{\infty} dx(T) \int_{x(t)}^{x(T)} \mathcal{D}x(s) \exp\left(-\frac{1}{2\sigma^2}\int_t^T ds\left(\left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)^2 + a^2x^2(s)\right)\right. \\ &- \left.\frac{a}{2\sigma^2}(x^2(T) - x^2(t))\right) \end{aligned} \tag{5.14}$$

The functional integrals over $x(s)$ can be evaluated using a familiar formula known in physics for the propagator of the harmonic oscillator (quoted in the appendix). From this formula one obtains for the numerator:

$$\begin{aligned} X &= \sqrt{\frac{a}{2\pi\sigma^2\sinh(a(T-t)}} \int_{-\infty}^{\infty} dx(T) \exp\left(\frac{\sigma^2}{2a^3}\left(e^{-a(T-t)} - 1 + a(T-t)\right)\right. \\ &- \frac{a}{2\sigma^2\sinh(a(T-t))}\left((x^2(T) + x^2(t))\cosh(a(T-t)) - 2x(T)x(t)\right. \\ &+ \left.2\left(e^{a(T-t)} - 1\right)(C(x(T)+x(t)) + C^2)\right) \\ &- \left.\frac{a}{2\sigma^2}(x^2(T) - x^2(t)) - b(T-t)\right) \end{aligned} \tag{5.15}$$

where

$$C = \frac{i\sigma^2}{2a}\int_0^{T-t} due^{-au}i = \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - 1\right) \tag{5.16}$$

Likewise one obtains an expression for the denominator:

$$\begin{aligned} Y &= \sqrt{\frac{a}{2\pi\sigma^2\sinh(a(T-t)}} \int_{-\infty}^{\infty} dx(T) \exp\left(-\frac{a}{2\sigma^2}(x^2(T) - x^2(t))\right. \\ &- \left.\frac{a}{2\sigma^2\sinh(a(T-t))}\left((x^2(T) + x^2(t))\cosh(a(T-t)) - 2x(T)x(t)\right)\right) \end{aligned} \tag{5.17}$$

It remains to perform the (Gaussian) integration with respect to $x(T)$. In agreement with standard notation (see e.g. Hull [6]), we define

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{5.18}$$

The relation to our function C is as follows:

$$C = -\frac{\sigma^2}{2a}B(t,T) \tag{5.19}$$

Collecting terms from the Gaussian integration in Eq.s (5.15) and (5.17) and substituting the results for X and Y in Eq.(5.12), one finally obtains:

$$P(t,T) = A(t,T)\exp\left(-B(t,T)r\right) \tag{5.20}$$

with

$$A(t,T) = \exp\left((-B(t,T) + T - t)\left(\frac{\sigma^2}{2a^2} - b\right) - \frac{\sigma^2}{4a}B^2(t,T)\right) \tag{5.21}$$

This result is in full agreement with an earlier result by Vasicek who uses the PDE method (see [9]). As we have successfully tested the method for a known result, it is justified to use it for further applications.

Concerning the functional integrations performed above, the question of existence of the functional integration measure $\mathcal{D}r(s)$ and $\mathcal{D}f(s)$ needs to be addressed. A full treatment of this issue is beyond the scope of this article (see e.g. [22]). Instead, we present a heuristic approach. Let us consider a functional integration $\int \mathcal{D}f(s)$ within the time interval $[t, T]$ which corresponds to fixing the functions $f(s)$ at times t and T to $f(t)$ and $f(T)$ respectively. We slice the time interval $[t, T]$ in N intervals of lenght τ such that $T - t = N\tau$. Thus we obtain a sequence of times $t_0 = t < t_1 < .. < t_i < t_{i+1} < .. < t_N = T$ with $i = 0..N$. From a heuristic viewpoint, an integration operator $\mathcal{D}f(s)$ may be written as a limit:

$$\int_{f(t)}^{f(T)} \mathcal{D}f(s) = \lim_{N \to \infty} \prod_{i=1}^{N-1} \int df(t_i) \tag{5.22}$$

The integrations on the r.h.s. are simple (Riemannian) integrations with respect to real variables $f(t_i)$. In fact, the r.h.s. of Eq.(5.22) is used for numerical evaluations of functional integrals [19]. In this sense, the sum over all paths taken by the function $f(s)$ from $f(t)$ to $f(T)$ is approximated by a walk with discrete steps in time. Of course, one has to make sure that after performing the limit $N \to \infty$ the results of functional integrations are independant of the way the time interval was discretized. Moreover, one has to take care of diverging normalization factors (appearing e.g. in section 4). Since all results we are interested in appear as expectation values and therefore as ratios of functional integrals, possibly diverging normalizations cancel (which is easy to show in a setting of discrete time). A mathematically rigorous treatment of functional integrals used for expectation values involves the Cameron-Martin-Girsanov theorem [22].

6 The bond option price as a path integral

As we propose the path integral approach in particular for the pricing of interest rate derivatives, let us demonstrate its usefulness for the example of bond options when short rates are again governed by Vasicek dynamics.

Specifically, let us consider the price of a European call at time 0 with strike k expiring at time t on zero bond with maturity T, with $T > t$:

$$c(0) = E_Q\left[e^{-\int_0^t ds r(s)} \max\left(P(t,T) - k; 0\right) | r(0) = r\right] \tag{6.23}$$

The function $\max(x; 0)$ gives the larger value of x or 0. For Vasicek dynamics the European call price can be determined analytically. The r.h.s. of Eq.(6.23) vanishes

unless $P(t,T) > k$. As in our case the zero bond price has the form

$$P(t,T) = A(t,T)e^{-B(t,T)r} \tag{6.24}$$

this condition can be rewritten as:

$$r(t) < \frac{1}{B(t,T)} \ln\left(\frac{A(t,T)}{k}\right) = \tilde{k} \tag{6.25}$$

The call price formula is then given by:

$$\begin{aligned} c(0) &= A(t,T)E_Q\left[e^{-\int_0^t dsr(s) - B(t,T)r} | r(t) < \tilde{k}\right] \\ &- kE_Q\left[e^{-\int_0^t dsr(s)} | r(t) < \tilde{k}\right] \end{aligned} \tag{6.26}$$

We now follow the steps outlined in the previous section. First, we substitute $x(s) = b - r(s)$. The condition $r(t) < \tilde{k}$ then translates into $x(t) > b - \tilde{k}$. The first expectation value on the r.h.s. of Eq.(6.26) is then given by:

$$E_Q\left[e^{-\int_0^t dsr(s) - B(t,T)r} | r(t) < \tilde{k}\right] = \frac{X}{Y} \tag{6.27}$$

with

$$\begin{aligned} X &= \int_{b-\tilde{k}}^{\infty} dx(t) \int_{x(0)}^{x(t)} \mathcal{D}x(s) \exp\left(-\frac{1}{2\sigma^2}\int_0^t ds \left(\left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)^2 + a^2x^2(s)\right)\right. \\ &- \left.\frac{a}{2\sigma^2}(x^2(t) - x^2(0)) - b(T-t) + \int_0^t dsx(s) - B(t,T)(b - x(t))\right) \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} Y &= \int_{-\infty}^{\infty} dx(t) \int_{x(0)}^{x(t)} \mathcal{D}x(s) \exp\left(-\frac{1}{2\sigma^2}\int_0^t ds \left(\left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)^2 + a^2x^2(s)\right)\right. \\ &- \left.\frac{a}{2\sigma^2}(x^2(t) - x^2(0))\right) \end{aligned} \tag{6.29}$$

The condition $x(t) > b - \tilde{k}$ enters into numerator X as a lower bound of integration with respect to $x(t)$. The next steps of integration are straightforward and follow along the lines of Eq.s(5.15) to (5.20). Useful relations are:

$$P(0,T) = E_Q\left[e^{-\int_0^t dsr(s)} P(t,T) | r(0) = r\right] \tag{6.30}$$

and

$$P(0,t) = E_Q\left[e^{-\int_0^t dsr(s)} | r(0) = r\right] \tag{6.31}$$

which follow form the no-arbitrage condition Eq.(7.37) to be discussed later. The bond prices $P(0,T)$ and $P(0,t)$ make up for the two terms on the r.h.s. of Eq.(6.26) up to the factors $N(h)$ and $kN(h-\sigma_P)$ which result form the bounded integrations with respect to $x(t)$ (see Eq.(6.28). In fact, we rederive the Jamshidian's formula for a European call on a zero bond [18]:

$$c(0) = P(0,T)N(h) - kP(0,t)N(h-\sigma_P) \tag{6.32}$$

where $N(x)$ is a cumulative normal distribution and

$$h = \frac{1}{\sigma_P} \ln\left(\frac{P(0,T)}{kP(0,t)}\right) + \frac{\sigma_P}{2} \tag{6.33}$$

with

$$\sigma_P = \frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right)\sqrt{\frac{1-e^{-2at}}{2a}} \tag{6.34}$$

For completeness we state the formula for a put which follows from put-call parity [6]:

$$p(0) = kP(0,t)N(-h+\sigma_P) - P(0,T)N(-h) \tag{6.35}$$

The analytical tractability of the bond option formulas for Vasicek dynamics is of course dependant on the simple form of the equation for the bond price itself. In general, this simple form remains true for so-called *affine* models [9] if one abstracts from the specific expressions given for $A(t,T)$ and $B(t,T)$. It allows for a simple translation of the condition on the terminal bond price into a condition on the terminal short rate. The latter can then be easily implemented as an integration boundary in the evaluation of the expectation value.

7 Models for the short rate in a new guise

Looking at the conditional probability density for the Vasicek model, Eq.(4.9), we can take the negative argument of the exponential function as a starting point to state models for the short rate in a different form. Using an analogy to physics, we call it the "Hamiltonian" for the short rate which reads in the case of Vasicek's model as follows:

$$H_{Vasicek} = \frac{1}{2\sigma^2}\int ds \left(\frac{\mathrm{d}r(s)}{\mathrm{d}s} - a(b-r(s))\right)^2 \tag{7.36}$$

New models for the short rate can be given by stating a different function H or a different probability density functional. Of course, these new models must be arbitrage-free in complete markets in the sense of [20]. In order to do so [9], they have to fulfil the following implicit constraint ($t \leq \tau \leq T$):

$$P(t,T) = E_Q\left[e^{-\int_t^\tau dsr(s)} P(\tau,T) | r(t) = r\right] \tag{7.37}$$

The Hamiltonian enters implicitly into the constraint in calculation of $P(t,T)$ on the l.h.s. and in the calculation of $P(\tau,T)$ and the evaluation of the expectation value on the r.h.s. of the equation. Eq.(7.37) is known to hold (and can be verified explicitly using the path integral) for Vasicek dynamics.

It would be desirable to derive a simple criterion for the class of functions H or the probability density functionals that are allowed by the no-arbitrage condition Eq.(7.37). In particular, it would be challenging to consider probability density functionals which are not derived from known short rate processes.

8 The path integral for general short rate dynamics

After having introduced the path integral approach for Vasicek dynamics, we generalize the approach to any model for the short rate which can be cast in the following form:

$$dr(t) = \rho(r(t),t)dt + \sigma(r(t),t)dz(t) \tag{8.38}$$

After following the procedure of section 4, the equivalent "Hamiltonian" in the path integral framwork is obtained:

$$H_{general} = \frac{1}{2}\int ds \frac{1}{\sigma(r(s),s)^2}\left(\frac{\mathrm{d}r(s)}{\mathrm{d}s} - \rho(r(s),s)\right)^2 + \int ds\phi(s)\sigma(r(s),s)\phi^*(s) \tag{8.39}$$

The first term in Eq.(8.39) is an obvious generalization of Eq.(7.36), the Hamiltonian of the Vasicek model. The second term, however, arises whenever $\sigma(r(s),s)$ is not constant, but depends on the short rate level $r(s)$ and/or time s. The functions $\phi(s)$ and $\phi^*(s)$ (the complex conjugate of $\phi(s)$) are random fields which need to be summed over in addition to the functional integration with respect to the short rate. In fact, every claim on a payoff X at time T is valued at time t according to:

$$V(t) = E_Q\left[e^{-\int_t^T dsr(s)}X|r(t)=r\right] \tag{8.40}$$

where the conditional expectation value is given by:

$$E_Q\left[Y|r(t)=r\right] = \frac{\int_{-\infty}^{\infty} dr(T)\int_{r(t)}^{r(T)}\mathcal{D}r(s)\int\mathcal{D}\phi(s)\int\mathcal{D}\phi^*(s)\exp\left(-H_{general}\right)Y}{\int_{-\infty}^{\infty} dr(T)\int_{r(t)}^{r(T)}\mathcal{D}r(s)\int\mathcal{D}\phi(s)\int\mathcal{D}\phi^*(s)\exp\left(-H_{general}\right)} \tag{8.41}$$

The integration boundaries for the random fields $\phi(s)$ and $\phi^*(s)$ have been suppressed. In fact, in a discretized form of the functional integral for a time interval $[t,T]$, the fields ϕ_i and ϕ_i^* for $i = 0..N$ for given points in time $t_i = t + i\tau$, where $N\tau = T - t$, are defined. The integration with respect to the in general complex fields ϕ_i and ϕ_i^* is mapped to an integration with respect to real and imaginary parts

$Re(\phi_i)$, $Im(\phi_i)$ respectively, with the integration boundaries ranging from $-\infty$ to ∞ [19]. At the end of the calculation the limit $N \to \infty$ and $\tau \to 0$ with $N\tau = T - t$ finite is performed.

The functional integrations in Eq.(8.41) can not be carried out exactly, but need to be performed on a lattice using a computer. However, this task can be accomplished in principle for any short rate model specified by a particular set $\rho(r(t), t)$ and $\sigma(r(t), t)$. This is the subject of future work.

9 Conclusion

We have presented a new approach to the pricing of interest rate derivatives, based on the path integral well developed in theoretical physics. It was tested for the simple case of a zero bond and a bond option in a world where the short rate is governed by Vasicek dynamics. The familiar results of Vasicek and Jamshidian were recovered. The approach was generalized to arbitrary short rate models given as a stochastic differential equation. Other claims can be readily priced within the new framework. However, analytical solutions are not always available.

Complementing numerical methods for pricing derivatives which take PDE's as their starting point such as finite elements, the new approach can be put to use in numerical applications. One possible choice is the direct computation of the path integral on a lattice. Another application would be in so-called path integral Monte Carlo simulations, which are also well established in physics [21]. Thus, the path integral approach might prove to become a powerful tool derivatives pricing and financial engineering.

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Appendix: The generating function for the harmonic oscillator

The following path integral is evaluated in [5]:

$$F[f; x, x'] = \int_{x(0)=x}^{x(U)=x'} \mathcal{D}x(u) \exp\left(-\int_0^U du \left(\frac{m}{2}\dot{x}^2 + \frac{m\omega^2}{2}x^2 + if(u)x(u)\right)\right) \quad (9.42)$$

The r.h.s. of Eq.(9.42) is given by:

$$F[f; x, x'] = \sqrt{m\omega/(2\pi \sinh(\omega U))}e^{-\Phi} \quad (9.43)$$

where

$$\Phi = \frac{1}{4m\omega}\int_0^U du\int_0^U du' e^{-\omega|u-u'|}f(u)f(u') + \frac{m\omega}{2\sinh(\omega U)}\Big[(x^2+x'^2)\cosh(\omega U) - 2xx' + 2A(xe^{\omega U}-x') + 2B(x'e^{\omega U}-x) + (A^2+B^2)e^{\omega U} - 2AB\Big] \tag{9.44}$$

where

$$\begin{aligned} A &= \frac{i}{2m\omega}\int_0^U du e^{-\omega u} f(u) \\ B &= \frac{i}{2m\omega}\int_0^U du e^{-\omega(U-u)} f(u) \end{aligned} \tag{9.45}$$

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